Journal of Engineering Mathematics, Vol. 15, No. 3, July 1981

Printed in the Netherlands

Hodograph transformations and solutions in constantly inclined MHD plane flows

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(Received August 26, 1980)

SUMMARY

In this work we develop a technique, based on the hodograph method, for the study of steady plane, viscous, incompressible constantly inclined MHD flows.

The equation describing the diffusion of magnetic field is used to write the magnetic field vector in terms of the velocity vector field and the angle between the two vector fields. The hodograph method (and its modifications) is applied to reduce the problem to that of determining the Legendre transform of the stream function. The resulting partial differential equation is studied for several flow problems to illustrate the advantages of the theory.

This paper also employs a similar approach as the above to study flows in the magnetograph plane.

1. Introduction

MHD plane flows are said to be constantly inclined if the angle between the velocity and the magnetic field vectors is constant throughout the flow region. Two special classes of these flows are aligned, or parallel, flows and crossed, or orthogonal, flows; these have been extensively studied over the past two decades. Not much work seems to have been done for those constantly inclined flows which are not necessarily aligned or crossed. These general constantly inclined flows were first investigated by Waterhouse and Kingston [1] when the fluid is inviscid and incompressible with infinite electrical conductivity. Chandna and Garg [2], Chandna and Toews [3] obtained several geometric results for such flows when the fluid is viscous incompressible and perfectly conducting.

In the present paper the work in viscous, incompressible flows is extended with the objective of obtaining some exact solutions. We employ the hodograph transformation, one of the strong analytic methods, to find solutions. An excellent survey of this method, with applications to numerous non-linear problems, has been given by Ames [4]. The hodograph method has been applied to parallel compressible MHD flows by Smith [5] and to viscous incompressible orthogonal MHD flows by Chandna and Garg [6].

The plan of the paper is as follows: in Section 2 the basic flow equations are cast into a convenient form for this work and Section 3 deals with the transformation of the equations to the hodograph plane. Solutions in the hodograph plane are found in Section 4. In Section 5 we transform the flow equations to the magnetograph plane and discuss their solutions.

0022-0833/81/030211-10$00.20/0

2. Equations of motion

The steady, plane flow of a viscous, incompressible fluid of infinite electrical conductivity is governed by the following system of equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial p}{\partial x} = \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \mu H_2\left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right), \tag{2}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y} = \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \mu H_1\left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right), \tag{3}$$

$$u H_2 - v H_1 = k, \tag{4}$$

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0, \tag{5}$$

where u, v are the components of the velocity field $\boldsymbol{V}$, H_1, H_2 the components of the magnetic vector field $\boldsymbol{H}$ and p is the pressure function; all being functions of x, y. In this system ρ, η, μ, are respectively the constant fluid density, the constant coefficient of viscosity, the constant magnetic permeability. Furthermore, k is an arbitrary constant of integration obtained from the diffusion equation $\operatorname{curl}(\boldsymbol{V} \times \boldsymbol{H}) = \boldsymbol{0}$; k is zero for aligned flows and non-zero in the case of non-aligned flows.

Introducing the functions

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \quad j = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}, \quad h = \frac{1}{2}\rho q^2 + p, \tag{6}$$

where $q^2 = u^2 + v^2$, the system of equations (1) to (5) is replaced by the following system:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \text{(continuity)}$$

$$\eta\frac{\partial \omega}{\partial y} - \rho v \omega + \mu j H_2 = -\frac{\partial h}{\partial x}, \qquad \text{(linear momentum)}$$

$$\eta\frac{\partial \omega}{\partial x} - \rho u \omega + \mu j H_1 = \frac{\partial h}{\partial y},$$

$$u H_2 - v H_1 = k, \qquad \text{(diffusion)} \tag{7}$$

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0, \qquad \text{(solenoidal)}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega, \qquad \text{(vorticity)}$$

$$\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = j, \qquad \text{(current density)}$$

of seven non-linear partial differential equations in seven unknowns $u, v, H_1, H_2, \omega, j$ and h as functions of x, y. System (7) has the advantage of being a system of first order whereas equations (1) to (5) form a second order system. Martin [7] has successfully used such a reduction of order from two to one to study viscous non-MHD flows.

We now consider constantly inclined plane flows and let α_0 denote the constant non-zero angle between $\boldsymbol{V}$ and $\boldsymbol{H}$. The vector and scalar products of $\boldsymbol{V}$ and $\boldsymbol{H}$, using the diffusion equation from (7), yield

$$\begin{aligned} u H_2 - v H_1 &= q H \sin \alpha_0 = k, \\ u H_1 + v H_2 &= q H \cos \alpha_0 = k \cot \alpha_0, \end{aligned} \tag{8}$$

where $H = \sqrt{H_1^2 + H_2^2}$. Considering these as two linear algebraic equations in the unknowns H_1, H_2, we solve these to express H_1 and H_2 in terms of u and v; i.e.

$$H_1 = \frac{k}{q^2}(cu - v), \quad H_2 = \frac{k}{q^2}(cv + u), \tag{9}$$

where $c = \cot \alpha_0$ is a known constant for a prescribed constantly inclined non-aligned flow.

Alternatively, one can solve (8) for u and v in terms of H_1 and H_2 to get

$$u = \frac{k}{H^2}(cH_1 + H_2), \quad v = \frac{k}{H^2}(cH_2 - H_1). \tag{10}$$

We now distinguish between two types of approaches. First, equation (9) can be employed to eliminate functions H_1 and H_2 from the system of equations (7). The unknown functions, to be determined, will then be u, v, h, ω and j. Secondly, one can eliminate u and v from equations (7) by using equations (10). One then obtains a system of equations to be solved for H_1, H_2, h, ω and j as functions of x, y. The first approach leads us to the study of flows, after hodograph transformations, in the hodograph plane. Likewise, the second approach leads to the study in the magnetograph plane.

3. Study of flows in the hodograph plane

Taking the first approach, functions H_1 and H_2 are eliminated from equations (7), by using equations (9), to yield the following system of equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{11}$$

$$\eta \frac{\partial \omega}{\partial y} - \rho v \omega + \frac{\mu k}{q^2}(cv + u) j = -\frac{\partial h}{\partial x}, \tag{12}$$

$$\eta\frac{\partial\omega}{\partial x}-\rho u\omega+\frac{\mu k}{q^2}(cu-v)j=\frac{\partial h}{\partial y}, \tag{13}$$

$$(v^2-u^2-2cuv)\left\{\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right\}+(cv^2-cu^2+2uv)\left\{\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right\}=0, \tag{14}$$

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\omega, \tag{15}$$

$$\frac{\partial}{\partial x}\left\{\frac{cv+u}{q^2}\right\}-\frac{\partial}{\partial y}\left\{\frac{cu-v}{q^2}\right\}=\frac{j}{k}. \tag{16}$$

We notice that the two equations (11) and (14) constitute a system of two equations in two unknown functions $u(x, y)$ and $v(x, y)$. Having found u and v, functions $\omega(x, y)$ and $j(x, y)$ are determined from equations (15) and (16) respectively. However, equations (12) and (13) force restrictions on those possible solutions for u, v, ω and j since the functions $h(x, y)$, to be obtained from (12) and (13), must satisfy the integrability condition $\partial^2 h/\partial x\partial y=\partial^2 h/\partial y\partial x$. Once a solution of this system is known, the pressure function and the magnetic vector field are obtained from the definition of h and equations (9) respectively.

Let the flow variables $u(x, y)$, $v(x, y)$ be such that, in the flow region under consideration, the Jacobian $J=\partial(u, v)/\partial(x, y)$ satisfies $0<|J|<\infty$. In such a case we may consider x and y as functions of u and v such that the following relations hold true:

$$\frac{\partial u}{\partial x}=J\frac{\partial y}{\partial v},\ \frac{\partial u}{\partial y}=-J\frac{\partial x}{\partial v},\ \frac{\partial v}{\partial x}=-J\frac{\partial y}{\partial u},\ \frac{\partial v}{\partial y}=J\frac{\partial x}{\partial u}. \tag{17}$$

Employing the transformation equations (17) in (11) to (16), the new transformed system of equations in the hodograph plane is:

$$\frac{\partial x}{\partial u}+\frac{\partial y}{\partial v}=0, \tag{18}$$

$$\eta J\frac{\partial(x,\omega)}{\partial(u,v)}-\rho v\omega+\frac{\mu k}{q^2}(cv+u)j=-J\frac{\partial(h,y)}{\partial(u,v)}, \tag{19}$$

$$\eta J\frac{\partial(\omega,y)}{\partial(u,v)}-\rho u\omega+\frac{\mu k}{q^2}(cu-v)j=J\frac{\partial(x,h)}{\partial(u,v)}, \tag{20}$$

$$(cu^2-cv^2-2uv)\left\{\frac{\partial x}{\partial u}-\frac{\partial y}{\partial v}\right\}+(u^2-v^2+2cuv)\left\{\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\right\}=0, \tag{21}$$

$$J\left\{\frac{\partial x}{\partial v}-\frac{\partial y}{\partial u}\right\}=\omega, \tag{22}$$

$$J\left\{\frac{\partial\left(\frac{u+cv}{q^2},y\right)}{\partial(u,v)}-\frac{\partial\left(x,\frac{cu-v}{q^2}\right)}{\partial(u,v)}\right\}=\frac{j}{k}, \tag{23}$$

where $J = J(u, v)$ is given by

$$J = \left[\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right]^{-1}. \tag{24}$$

Equation (18) implies the existence of a function $\Psi(u, v)$, called the Legendre transform of the streamfunction $\psi(x, y)$ defined by the continuity equation, such that [4]

$$\frac{\partial \Psi}{\partial u} = -y, \quad \frac{\partial \Psi}{\partial v} = x, \tag{25}$$

and

$$\Psi(u, v) = vx - uy + \psi(x, y). \tag{26}$$

We now eliminate $x(u, v)$ and $y(u, v)$ from equations (18) to (24) by introducing $\Psi(u, v)$ as defined by (25) or (26). Equation (18) is identically satisfied and equations (19) to (24) are replaced by the system of equations

$$\eta J \frac{\partial\left(\frac{\partial \Psi}{\partial v}, \omega\right)}{\partial(u, v)} - \rho v \omega + \frac{\mu k}{q^2}\left(cv + u\right) j = J \frac{\partial\left(h, \frac{\partial \Psi}{\partial u}\right)}{\partial(u, v)}, \tag{27}$$

$$\eta J \frac{\partial\left(\omega, \frac{\partial \Psi}{\partial u}\right)}{\partial(u, v)} + \rho u \omega - \frac{\mu k}{q^2}\left(cu - v\right) j = -J \frac{\partial\left(\frac{\partial \Psi}{\partial v}, h\right)}{\partial(u, v)}, \tag{28}$$

$$\left(v^2 - u^2 - 2cuv\right)\frac{\partial^2 \Psi}{\partial u^2} + \left(2cu^2 - 2cv^2 - 4uv\right)\frac{\partial^2 \Psi}{\partial u \partial v} + \left(2cuv + u^2 - v^2\right)\frac{\partial^2 \Psi}{\partial v^2} = 0, \tag{29}$$

$$J\left(\frac{\partial^2 \Psi}{\partial u^2} + \frac{\partial^2 \Psi}{\partial v^2}\right) = \omega, \tag{30}$$

$$J\left[\frac{\partial\left(\frac{u + cv}{q^2}, \frac{\partial \Psi}{\partial u}\right)}{\partial(u, v)} + \frac{\partial\left(\frac{\partial \Psi}{\partial v}, \frac{cu - v}{q^2}\right)}{\partial(u, v)}\right] = -\frac{j}{k}, \tag{31}$$

$$J = \left[\frac{\partial^2 \Psi}{\partial u^2}\frac{\partial^2 \Psi}{\partial v^2} - \left(\frac{\partial^2 \Psi}{\partial u \partial v}\right)^2\right]^{-1}, \tag{32}$$

in six unknowns Ψ, h, ω, j and J as functions of u, v. Finally, we introduce the polar co-ordinates (q, θ) in hodograph plane, i.e. (u, v)-plane, defined by the relations

$$u = q\cos\theta, \quad v = q\sin\theta, \tag{33}$$

where θ is the inclination of vector field V, and note that

$$\frac{\partial}{\partial u} = \cos\theta \frac{\partial}{\partial q} - \frac{\sin\theta}{q}\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial v} = \sin\theta \frac{\partial}{\partial q} + \frac{\cos\theta}{q}\frac{\partial}{\partial \theta}$$

and

$$\frac{\partial(F,G)}{\partial(u,v)} = \frac{\partial(F,G)}{\partial(q,\theta)} \cdot \frac{\partial(q,\theta)}{\partial(u,v)} = \frac{1}{q}\frac{\partial(F,G)}{\partial(q,\theta)} \tag{34}$$

for any continuously differentiable functions $F(u, v)$ and $G(u, v)$.

Transformation of equation (29), by use of equations (33) and (34), to the new independent variables q, θ (after considerable simplification) yields a second-order linear partial differential equation satisfied by $\bar{\Psi}(q, \theta) = \Psi(q\cos\theta, q\sin\theta)$ given as

$$\frac{\partial^2 \bar{\Psi}}{\partial q^2} - \frac{2c}{q}\frac{\partial^2 \bar{\Psi}}{\partial q \partial \theta} - \frac{1}{q^2}\frac{\partial^2 \bar{\Psi}}{\partial \theta^2} - \frac{1}{q}\frac{\partial \bar{\Psi}}{\partial q} + \frac{2c}{q^2}\frac{\partial \bar{\Psi}}{\partial \theta} = 0. \tag{35}$$

Furthermore, by using (33) and (34) in (30), (31) and (32), equations for ω, j and J as functions of q, θ are:

$$\omega = J\left[\frac{\partial^2 \bar{\Psi}}{\partial q^2} + \frac{1}{q^2}\frac{\partial^2 \bar{\Psi}}{\partial \theta^2} + \frac{1}{q}\frac{\partial \bar{\Psi}}{\partial q}\right], \tag{36}$$

$$j = \frac{kJ}{q^2}\left[c\frac{\partial^2 \bar{\Psi}}{\partial q^2} - \frac{c}{q^2}\left(\frac{\partial^2 \bar{\Psi}}{\partial \theta^2} + q\frac{\partial \bar{\Psi}}{\partial q}\right) + 2\frac{\partial}{\partial q}\left(\frac{1}{q}\frac{\partial \bar{\Psi}}{\partial \theta}\right)\right], \tag{37}$$

$$J = q^4\left[q^2 \frac{\partial^2 \bar{\Psi}}{\partial q^2}\left\{q\frac{\partial \bar{\Psi}}{\partial q} + \frac{\partial^2 \bar{\Psi}}{\partial \theta^2}\right\} - \left\{\frac{\partial \bar{\Psi}}{\partial \theta} - q\frac{\partial^2 \bar{\Psi}}{\partial \theta \partial q}\right\}^2\right]^{-1}. \tag{38}$$

It is important to note that not all solutions of (35) can define feasible flow configurations. Allowable forms of $\bar{\Psi}(q, \theta)$ are further restricted by the integrability condition on h, obtained through equations (27) and (28).

4. Solutions in the hodograph plane

A general solution of (35) seems to be impossible and we therefore examine some special forms of solutions.

(*a*) Assume a solution of (35) in the form

$$\bar{\Psi}(q, \theta) = \sum_{n=0}^{\infty} f_n(\theta) q^n.$$

Substituting this into (35) and equating like powers of q, we find that, for $c \neq 0$ (non-orthogonal flows),

$$\begin{aligned}\bar{\Psi}(q,\theta) &= A_0 e^{2c\theta} + B_0 + \{A_1 \cos\theta + B_1 \sin\theta\} q \\ &+ \{A_2 e^{-2c\theta} + B_2\} q^2 + \sum_{n=3}^{\infty} \{A_n e^{\lambda_n^+ \theta} + B_n e^{\lambda_n^- \theta}\} q^n,\end{aligned} \tag{39}$$

where

$$\lambda_n^{\pm} = -c(n-1) \pm \sqrt{(c^2+1)(n^2-2n)+c^2}$$

and A_j, B_j $(j \geqslant 0)$, are arbitrary constants. Various flow configurations are obtained by setting some of the coefficients in the series for $\bar{\Psi}(q,\theta)$ equal to zero.

(i) Vortex flow: Choosing $A_0 = B_0 = A_2 = 0$ and A_j, B_j for $j \geqslant 3$ all zero we attempt $\bar{\Psi}(q,\theta) = B_2 q^2 + (A_1 \cos\theta + B_1 \sin\theta)q$ where $B_2 \neq 0$. In this case the streamlines are given by

$$(x - B_1)^2 + (y + A_1)^2 = \text{constant}$$

and

$$\begin{aligned}&j = 0, \quad \omega = B_2^{-1}, \\ &p = \rho\{(x - B_1)^2 + (y + A_1)^2\}/8B_2^2 + \text{constant}, \\ &u = -(y + A_1)/2B_2, \quad v = (x - B_1)/2B_2.\end{aligned}$$

(ii) Radial flow: This type of flow requires that $\bar{\Psi}(q,\theta)$ be a function of θ only, hence, we try

$$\bar{\Psi}(q,\theta) = A_0 e^{2c\theta}.$$

From equations (36), (37) and (38)

$$J = -\frac{q^4 e^{-4c\theta}}{4c^2 A_0^2}, \quad \omega = -\frac{q^2 e^{-2c\theta}}{A_0}, \quad j = \frac{k(c^2+1)e^{-2c\theta}}{c^2 A_0}.$$

Using these, and eliminating $h(u, v)$ from equations (27) and (28) finally yields

$$\eta e^{-2c\theta}\{(5c^2 - 1)\cos 2\theta - 6c \sin 2\theta\} q^4 - \rho c A_0 q^4 - c\mu k^2 (c^2+1) = 0.$$

Since q and θ are independent variables, $\bar{\Psi}(q,\theta) = A_0 e^{2c\theta}$ is not a possible flow solution.

(iii) Other flows: Various other combinations of terms may be attempted, many leading to non-admissible forms for $\bar{\Psi}$, e.g.,

$$\bar{\Psi}(q,\theta) = \{A_2 e^{-2c\theta} + B_2\} q^2,$$

eventually leads to $A_2 = 0$ (example (i)) or $c = 0$ (orthogonal, vortex flow).

(*b*) Returning to equation (35) assume now a solution of the form

$$\bar{\Psi}(q, \theta) = F(q) + G(\theta).$$

Substituting this form into equation (35) and separating variables gives

$$\bar{\Psi}(q, \theta) = A \ln q + Bq^2 + \frac{A}{c}\theta + De^{2c\theta} + E \tag{40}$$

where A, B, D, E are arbitrary constants. Again, this solution is further restricted by equations (27) and (28).

(i) *Spiral flow:* Take $\bar{\Psi}(q, \theta) = A \ln q + \frac{A}{c}\theta$.

From equations (36), (37), (38), (27) and (28)

$$\omega = 0, \quad j = 2kc/A,$$

$$p = -\frac{\mu k^2 c^2}{A^2}(x^2 + y^2) - \frac{\rho A^2(c^2 + 1)}{2c^2(x^2 + y^2)} + \text{constant}.$$

To determine the streamlines we calculate

$$x = \frac{\partial \Psi}{\partial v} = \frac{A}{u^2 + v^2}\left\{\frac{u}{c} + v\right\}, \quad y = -\frac{\partial \Psi}{\partial u} = \frac{A}{u^2 + v^2}\left\{\frac{v}{c} - u\right\}.$$

Taking the ratio x/y and then solving for v/u, the streamlines are the solution of

$$\frac{dy}{dx} = \frac{v}{u} = \frac{y + cx}{x - cy}.$$

Hence the equation for the streamlines is

$$\ln (x^2 + y^2)^{1/2} = \frac{1}{c}\tan^{-1}\frac{y}{x} + \text{constant},$$

representing a spiral flow with constantly-inclined circular magnetic lines.

(ii) As a second example in this section we try

$$\bar{\Psi}(q, \theta) = A \ln q + Bq^2 + \frac{A}{c}\theta.$$

Then equations (36), (37) and (38) give

$$J = \frac{c^2 q^4}{4B^2c^2q^4 - (c^2 + 1)A^2},$$

$$\omega = 4BJ, \quad j = -\frac{2Ak(c^2 + 1)J}{cq^4}.$$

Using these, and eliminating $h(u, v)$ from equations (27) and (28) yields an expression of the form

$$\sum_{n=1}^{5} P_n q^{2n+1} = 0,$$

implying that $P_n \equiv 0$, $n = 1, \ldots, 5$. In particular $P_4 = 0$ implies

$$\eta A^3 B^4 = 0.$$

Since $A = 0$ or $B = 0$ lead to previously discussed examples, the interesting case here is the inviscid problem ($\eta = 0$). With $\eta = 0, P_i \equiv 0$ for $i = 1, 2, 3, 5$ all imply that

$$B = \frac{A}{2kc}\sqrt{\frac{\rho}{\mu}}\,.$$

Following the previous example we find that the streamlines are the solution curves of the O.D.E.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 - y^2 - 2cxy \pm \sqrt{r^4 - 8ABr^2 - 16A^2B^2/c^2}}{2\left(\dfrac{2AB}{c} + cy^2 - xy\right)}\,.$$

These streamlines are spirals, previously discussed by Waterhouse and Kingston [1] in their classification of inviscid incompressible flows. We see that these types of spirals cannot exist in the viscous case.

5. Study of flows in the magnetograph plane

Analogous to the work in Section 3, u and v can be eliminated from the governing system of equations by using (10). Introducing $\Phi(H_1, H_2)$, the Legendre transform of the magnetic flux function $\phi(x, y)$, by

$$\frac{\partial \Phi}{\partial H_1} = -y, \quad \frac{\partial \Phi}{\partial H_2} = x \tag{41}$$

and polar co-ordinates (H, β) in the magnetograph plane, a necessary condition for $\overline{\Phi}(H, \beta) = \Phi(H\cos\beta, H\sin\beta)$ to define an admissible flow solution is that it be a solution of

$$H^2\frac{\partial^2\overline{\Phi}}{\partial H^2} + 2cH\frac{\partial^2\overline{\Phi}}{\partial H\partial\beta} - \frac{\partial^2\overline{\Phi}}{\partial\beta^2} - H\frac{\partial\overline{\Phi}}{\partial H} - 2c\frac{\partial\overline{\Phi}}{\partial\beta} = 0. \tag{42}$$

This equation is the magnetograph counterpart of equation (35). As in Section 3, we find

$$j = J^*\left[\frac{\partial^2\overline{\Phi}}{\partial H^2} + \frac{1}{H^2}\frac{\partial^2\overline{\Phi}}{\partial\beta^2} + \frac{1}{H}\frac{\partial\overline{\Phi}}{\partial H}\right], \tag{43}$$

$$\omega = \frac{kJ^*}{H^2}\left[c\frac{\partial^2\bar{\Phi}}{\partial H^2} - \frac{c}{H^2}\left\{\frac{\partial^2\bar{\Phi}}{\partial\beta^2} + H\frac{\partial\bar{\Phi}}{\partial H}\right\} - \frac{2\partial}{\partial H}\left(\frac{1}{H}\frac{\partial\bar{\Phi}}{\partial\beta}\right)\right], \tag{44}$$

$$J^* = H^4\left[H^2\frac{\partial^2\bar{\Phi}}{\partial H^2}\left\{H\frac{\partial\bar{\Phi}}{\partial H} + \frac{\partial^2\bar{\Phi}}{\partial\beta^2}\right\} - \left\{\frac{\partial\bar{\Phi}}{\partial\beta} - H\frac{\partial^2\bar{\Phi}}{\partial\beta\partial H}\right\}^2\right]^{-1}. \tag{45}$$

The functions $\bar{\Phi}(H,\beta)$ satisfying equation (42) are again further restricted by an integrability condition on $h(H_1, H_2)$ imposed by the momentum equations.

Equation (42) is identical to equation (35) if we replace c in (42) by $-c$. Hence solutions of (42) can be immediately written down as done in section 4. We investigate two examples in the magnetograph plane.

(i) $\bar{\Phi}(H,\beta) = B_2H^2$. In this case the magnetic lines are concentric circles and the streamlines are spiral inclined at a constant angle α_0 to the magnetic lines. This example is the same as that of Section 4, example (b)(i).

(ii) $\bar{\Phi}(H,\beta) = A\ \ln H - \frac{A}{c}\beta$. Here we obtain the vortex flow described in the first example of Section 4. The magnetic lines are spiral and the streamlines are concentric circles.

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